Math 210A Lecture 10 Notes

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1 Images, Coimages, and Generating Sets

1.1 Images

Definition 1.1. The **image** $\mathrm{im}(f)$ of $f : A \to B$ is an object and a monomorphism $\iota : \mathrm{im}(f) \to B$ such that there exists $\pi : A \to \mathrm{im}(f)$ with $\pi \circ \iota$ and such that if $e : C \to B$ is a monomorphism and $g : A \to C$ is such that $e \circ g = f$, then there exists a unique morphism $\psi : \mathrm{im}(f) \to C$ such that $g \circ \psi = \iota$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi"] \arrow[ddr, "g"'] & & B \\
& \mathrm{im}(f) \arrow[ur, "\iota"] \arrow[d, dashed, "\psi"'] & \\
& C \arrow[uur, "e"'] &
\end{tikzcd}$$

Example 1.1. In Set, $f(A) = \mathrm{im}(f)$. Then $b \in F(A) \implies b = f(a)$ for some $a \in A$. Then $g(a) \in C$ is the unique element with $e(g(a)) = (a)$ because e is a monomorphism. So $\psi(f(a)) = g(a)$.

Proposition 1.1. *If $\mathcal{C}$ has equalizers, then $\pi : A \to \mathrm{im}(f)$ is an epimorphism.*

Proof. Suppose

$$\begin{tikzcd}
A \arrow[r, "\iota"] & \mathrm{im}(f) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & D
\end{tikzcd}$$

commutes. Then $\alpha \circ \pi = \beta \circ \pi$,

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "\pi"] \arrow[drr, "f"] & \mathrm{eq}(\alpha, \beta) \arrow[r, "c"] & \mathrm{im}(f) \arrow[d, "\iota"] \\
& & B
\end{tikzcd}$$

Then there is a unique $d : \mathrm{im}(f) \to \mathrm{eq}(\alpha, \beta)$, and $c \circ d = \mathrm{id}$ and $d \circ c = \mathrm{id}$ by uniqueness. So $(\mathrm{im}(f), \mathrm{id}_{\mathrm{im}(f)})$ equalizes

$$\begin{tikzcd} \mathrm{im}(f) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & D \end{tikzcd}$$

so $\alpha = \beta$. ☐

Suppose that in $\mathcal{C}$, every morphism factors through an equalizer and the category has finite limits and colimits. Then $\mathrm{im}(f)$ can be defined as the equalizer of the following diagram:

$$\begin{tikzcd} B \arrow[r, shift left, "\iota_1"] \arrow[r, shift right, "\iota_2"'] & B \amalg_A B \end{tikzcd}$$

We get the following diagram.

$$\begin{tikzcd}
& A \arrow[ddl, "f"'] \arrow[d, dashed, "\pi"] \arrow[ddr, "f"] & \\
& \mathrm{im}(f) \arrow[dl, "\iota"] \arrow[dr, "\iota"'] & \\
B \arrow[dr, "\iota_2"'] & & B \arrow[dl, "\iota_1"] \\
& B \amalg_A B &
\end{tikzcd}$$

1.2 Coimages

Definition 1.2. The **coimage** $\mathrm{im}(f)$ of $f : A \to B$ is an object and a monomorphism $\pi : A \to \mathrm{coim}(f)$ such that there exists $\iota : \mathrm{coim}(f) \to B$ such that $\iota \circ \pi$ and such that if $g : A \to C$ is an epinmorphism and $e : C \to B$ is such that $e \circ g = f$, then there exists a unique morphism $\theta : C \to \mathrm{coim}(f)$ such that $\theta \circ g = \pi$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi"] \arrow[ddr, "g"'] & & B \\
& \mathrm{coim}(f) \arrow[ur, "\iota"] \arrow[d, dashed, "\psi"'] & \\
& C \arrow[uur, "e"'] &
\end{tikzcd}$$

So $\iota \circ \theta \circ g = \iota \circ \pi = f = e \circ g$. Since g is an epimorphism, $i \circ \theta = e$.

How are the image and coimage related?

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr] \arrow[drr] & & & B \\
& \mathrm{im}(f) \arrow[urr] \arrow[r, dashed] & \mathrm{coim}(f) \arrow[ur] &
\end{tikzcd}$$

Definition 1.3. A morphism $f : A \to B$ is **strict** if $\mathrm{im}(f) \to \mathrm{coim}(f)$ is an isomorphism.

In Grp, Ring, Rmod, Set, and Top, $\mathrm{im}(f)$ is the set theoretic image. The coimages are quotient objects (of A).

Example 1.2. In Set, $\mathrm{coim}(f) = A/\sim$, where $a \sim a$; if $f(a) \sim f(a')$. All the morphisms are strict.

Example 1.3. In Gp, $\mathrm{coim}(f : C \to C') = G/\ker(f)$. $\mathrm{im}(f) \subseteq f(G) \leq G'$. So the image and coimage are isomorphic, which is the first isomorphism theorem.

Example 1.4. In Ring, let $\ker(f)$ be the category theoretic kernel. Then $\mathrm{coim}(f) = R/\ker(f) \xrightarrow{\sim} \mathrm{im}(f)$ by the first isomorphism theorem.

Example 1.5. In the category of left R-modules, morphisms are also strict.

1.3 Generating sets

Definition 1.4. Let $\Phi : \mathcal{C} \to \mathrm{Set}$ be a faithful functor, and let F be a left adjoint to Φ. Let $F_X = F(X)$ be the free object on X. If $X \xrightarrow{f} \Phi(A)$ for $A \in \mathcal{C}$, we get $\phi : F_X \to A$. Suppose $\mathrm{im}(\phi)$ exists. Then $\mathrm{im}(\phi)$ is called the **subobject of A generated by X**.

Example 1.6. In Gp, let $X \subseteq G$. Then $\langle X \rangle$ is the subgroup of G generated by X. This is $\mathrm{im}(\phi : T_X \to G)$, where $\phi(x_1^{n_1} \cdots x_r^{x_r}) = x_1^{n_1} \cdots x_r^{n_r}$. So this is $\{x_1^{n_1} \cdots x_r^{n_r} : x_1 \in X, n_i \in \mathbb{Z}, 1 \leq i \leq r, r \geq 0\}$. We claim that $\langle X \rangle$ is the smallest subgroup of G containing X, or equivalently, the intersection of all subgroups of G containing X. Indeed, this is a subgroup of G containing X, and any subgroup of G containing X must contain these words, since it must be closed under products.

Example 1.7. In Rmod, if $X \subseteq A$, $R \cdot X = \{\sum_{i=1}^n r_i x_i : r_i \in R, x_i \in X, 1 \leq i \leq n, n \geq 0\}$. So $F_X = \bigoplus_{x \in X} R_x \xrightarrow{\phi} A$, where $\phi(r \cdot x) = rx \in A$.

Example 1.8. In the category of (R, S)-bimodules, $RXS = \{\sum_{i=1}^n r_i x_i s_i : r_i \in R, s_i \in S, x_i \in X, 1 \leq i \leq n, n \geq 0\}$. If we have the set of formal sums $RxS = \{\sum_{i=1}^n r_i x s_i : r_i \in R, s_i \in S, 1 \leq i \leq n, n \geq 0\}$ with $(r + r')xs = rxs + r'xs$ and $rx(s + s') = rxs + rxs'$, then the free object is $\bigoplus_{x \in X} RxS$.

Ideals uses (R, R)-subbimodules of R generated by $X \subseteq R$.

Definition 1.5. The **ideal generated by** X is $(X) = \{\sum_{i=1}^{n} r_i x_i r_i' : r_i, r_i' \in R, x_i \in X\}$. If $X = \{x_1, \ldots, x_n\}$, then we write $(x_1, \ldots, x_n)$.

Remark 1.1. Even if $X = \{x\}$, we still need to take sums to get (x).